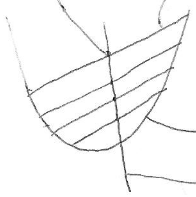
midpoints



Chords parallel to a given vector

a second-degree curve : $\underbrace{Ax^2+Bxy+Cy^2}_{\text{Quadratic form}}+\underbrace{Dx+Ey}_{\text{Linear form}}+F=0$

locus (轨迹) of the midpoints of the chords is a straight line

Parabola : $y=ax^2+bx+c$ (抛物线)

Circle : $(x-h)^2+(y-k)^2=r^2$

Ellipse : $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Hyperbola : $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ 双曲线

$\underbrace{A(x,x)}_{\text{Quadratic form}}+\underbrace{L(x)}_{\text{Linear form}}+C=0$ 证明 locus 是 straight line : 线性.

Bilinear forms : linear numerical functions of two vector arguments.
$A(x,y)$

当 $x=y$ 时 Bilinear form 变成 Quadratic form.

Symmetric bilinear forms : $A(x,y)=A(y,x)$.

One of the basic problems of plane analytic geometry is to reduce the general equation of a second-degree curve to canonical form by transforming to a new coordinate system.

$$Ax^2+\underbrace{2Bxy}_{\text{对称}}+Cy^2=D.$$

$Q(x)=x^TAx$

为什么通常只考虑 A symmetric? 因为非对称部分会抵消.

$$A=\frac{A+A^T}{2}+\frac{A-A^T}{2}$$

$$x^T(A-A^T)x=-x^T(A^T-A)x=-x^T(A-A^T)^Tx=-\underbrace{\left(x^T(A-A^T)x\right)}_{\text{常数 1-D 一维}}^T$$

所以 $x^T(A-A^T)x=0$

$$Q(x)=x^TAx=\frac{1}{2}x^T\underbrace{(A+A^T)}_{\text{Symmetric}}x$$

A is called the matrix of the quadratic form.

定义

- positive definite : $Q(x) > 0$ for all $x \neq 0$.
- negative definite : $Q(x) < 0$ for all $x \neq 0$
- indefinite : $Q(x)$ assumes both positive and negative values

$Q(x) \geq 0$ positive semidefinite.

$Q(x) \leq 0$ negative semidefinite.

positive definite $\Leftrightarrow$ eigenvalues of A are all positive. } Theorem

$\boxed{\text{Principal Axes}}$ Theorem : there is an orthogonal change of variable, $x = Py$ that transforms the quadratic form x^TAx into a quadratic form y^TDy with no cross-product term.

x, y : change of variable

要求 P 可逆. Change 的目的是消除 cross product.

columns of P : basis.

y is the coordinate vector. (在新的 basis 下，对应的新的坐标)

(y is the coordinate vector of x relative to the basis of R^n determined by the columns of P). 这里可以展开讲 Coordinate System.

$$x^TAx = (Py)^TAPy = y^T\underbrace{(P^TAP)}y$$

→ New matrix of the quadratic form is P^TAP.

→ D : Diagonalization of Symmetric Matrices.

变成 y^TDy → diagonal matrix. 因此，无 cross product terms

Theorem : An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

$$A = PDP^{-1} = PDP^T$$

研究 matrix diagonalization 的用途之一是消除 quadratic forms 的 cross-product terms.

$$P^TAP = P^TPDP^{-1}P = D \qquad (P^T = P^{-1})$$

→ P is an orthogonal matrix.